

# Yangian and Applications

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## Abstract

In this paper, the Yangian relations are tremendously simplified for Yangians associated to  $SU(2)$ ,  $SU(3)$ ,  $SO(5)$  and  $SO(6)$  based on RTT relations that much benefit the realization of Yangian in physics. The physical meaning and some applications of Yangian have been shown.

## 1 Introduction

Yangian was presented by Drinfel'd ([1-3]) twenty years ago. It receives more attention for the following reasons. It is related to the rational solution of Yang-Baxter equation and the RTT relation. It is a simple extension of Lie algebras and the representation theory of  $Y(SU(2))$  has been given. Some physical models, say, two component nonlinear Schrodinger equation, Haldane-Shastry model and 1-dimensional Hubbard chain do have Yangian symmetry. Yangian may be viewed as the consequence of a "bi-spin" system. How to understand the physical meaning of Yangian is an interesting topic. In this paper, there is nothing with mathematics. Rather, we try to use the language of quantum mechanics and Lie algebraic knowledge to show the effects of Yangian.

## 2 Yangian and RTT Relations

Let  $\mathcal{G}$  be a complex simple Lie algebra. The Yangian algebra  $Y(\mathcal{G})$  associated to  $\mathcal{G}$  was given as follows ([1-3]). For a given set of Lie algebraic generators  $I_\mu$  of  $\mathcal{G}$  the new generators  $J_\nu$  were introduced to satisfy

$$[I_\lambda, I_\mu] = C_{\lambda\mu\nu} I_\nu, \quad C_{\lambda\mu\nu} \text{ are structural constants;} \quad (2.0.1)$$

$$[I_\lambda, J_\mu] = C_{\lambda\mu\nu} J_\nu; \quad (2.0.2)$$

and, for  $\mathcal{G} \neq sl(2)$ :

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = a_{\lambda\mu\nu\alpha\beta\gamma} \{I_\alpha, I_\beta, I_\gamma\}, \quad (2.0.3)$$

where

$$a_{\lambda\mu\nu\alpha\beta\gamma} = \frac{1}{4!} C_{\lambda\alpha\sigma} C_{\mu\beta\tau} C_{\nu\gamma\rho} C_{\sigma\tau\rho}, \quad (2.0.4)$$

$$\{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k, \quad (\text{symmetric summation}); \quad (2.0.5)$$

or for  $\mathcal{G} = sl(2)$ :

$$\begin{aligned} & [[J_\lambda, J_\mu], [I_\sigma, J_\tau]] + [[J_\sigma, J_\tau], [I_\lambda, J_\mu]] \\ &= (a_{\lambda\mu\nu\alpha\beta\gamma} C_{\sigma\tau\nu} + a_{\sigma\tau\nu\alpha\beta\gamma} C_{\lambda\mu\nu}) \{I_\alpha, I_\beta, J_\gamma\}. \end{aligned} \quad (2.0.6)$$

When  $C_{\lambda\mu\nu} = i\varepsilon_{\lambda\mu\nu} (\lambda, \mu, \nu = 1, 2, 3)$ , equation (2.0.3) is identically satisfied from the Jacobian identities. Besides the commutation relations there are co-products as follows.

$$\Delta(I_\lambda) = I_\lambda \otimes 1 + 1 \otimes I_\lambda; \quad (2.0.7)$$

$$\Delta(J_\lambda) = J_\lambda \otimes 1 + 1 \otimes J_\lambda + \frac{1}{2} C_{\lambda\mu\nu} I_\mu \otimes I_\nu. \quad (2.0.8)$$

Further, the Yangian can be derived through RTT relations where  $R$  is a rational solution of Yang-Baxter equation (YBE) ([1-12]).

After lengthy calculations, we found the independent relations for  $Y(SU(2))$ ,  $Y(SU(3))$ ,  $Y(SO(5))$  and  $Y(SO(6))$  by expanding the RTT relations and also checked through equations (2.0.1)-(2.0.3) and (2.0.6) by substituting the structural constants ([13-17]), where RTT relation (Faddeev, Reshetikhin, Takhtajan — RFT [18]) satisfies

$$\check{R}(u-v)(T(u) \otimes 1)(1 \otimes T(v)) = (1 \otimes T(v))(T(u) \otimes 1)\check{R}(u-v). \quad (2.0.9)$$

## 2.1 $Y(SU(2))$

Let  $P_{12}$  be the permutation. Setting

$$\check{R}_{12}(u) = PR_{12}(u) = uP_{12} + I; \quad (2.1.1)$$

$$\begin{aligned} T(u) &= I + \sum_{n=1}^{\infty} u^{-n} \begin{bmatrix} T_{11}^{(n)} & T_{12}^{(n)} \\ T_{21}^{(n)} & T_{22}^{(n)} \end{bmatrix} \\ &= I + \sum_{n=1}^{\infty} u^{-n} \begin{bmatrix} \frac{1}{2}(T_0^{(n)} + T_3^{(n)}), & T_+^{(n)} \\ T_-^{(n)}, & \frac{1}{2}(T_0^{(n)} - T_3^{(n)}) \end{bmatrix}, \end{aligned} \quad (2.1.2)$$

and substituting the  $T(u)$  into RTT relation it turns out that only

$$I_{\pm} = T_{\pm}^{(1)}, I_3 = \frac{1}{2}T_3^{(1)}; \quad (2.1.3)$$

$$J_{\pm} = T_{\pm}^{(2)}, J_3 = \frac{1}{2}T_3^{(2)} \quad (2.1.4)$$

are independent ones. The quantum determinant

$$\det T(u) = T_{11}(u)T_{22}(u-1) - T_{12}(u)T_{21}(u-1) = C_0 + \sum_{n=1}^{\infty} u^{-n}C_n \quad (2.1.5)$$

gives

$$C_0 = 1, \quad C_1 = T_0^{(1)} = \text{tr} T^{(1)}, \quad (2.1.6)$$

$$C_2 = T_0^{(2)} - \mathbf{I}^2 + T_0^{(1)}(1 + \frac{1}{2}T_0^{(1)}), \quad \dots, \quad (2.1.7)$$

The independent commutation relations of  $Y(SU(2))$  are:

$$[I_{\lambda}, I_{\mu}] = i\epsilon_{\lambda\mu\nu}I_{\nu} \quad (\lambda, \mu, \nu = 1, 2, 3); \quad (2.1.8)$$

$$[I_{\lambda}, J_{\mu}] = i\epsilon_{\lambda\mu\nu}J_{\nu}; \quad (2.1.9)$$

and  $(A_{\pm} = A_1 \pm iA_2)$

$$[J_3, [J_+, J_-]] = (J_-J_+ - I_-J_+)I_3 \quad (2.1.10)$$

that can be checked to generate all of relations of equations (2.0.1), (2.0.2) and (2.0.6) with the help of Jacobi identities.

The co-product is given through (RFT) as

$$\Delta T_{ab} = \sum_c T_{ac} \otimes T_{cb}. \quad (2.1.11)$$

The simplest realization of  $Y(SU(2))$  is

$$\mathbf{I} = \sum_{i=1}^N \mathbf{I}_i \quad (i : \text{lattice indices}), \quad (2.1.12)$$

$$\mathbf{J} = \sum_{i=1}^N \mu_i \mathbf{I}_i + \sum_{i < j}^N W_{ij} \mathbf{I}_i \times \mathbf{I}_j, \quad (2.1.13)$$

where

$$W_{ij} = \begin{cases} 1 & i < j \\ 0 & i = j \\ -1 & i > j \end{cases} \quad (\text{for any representation of } SU(2)) \quad (2.1.14)$$

or

$$W_{jk} = i \cot \frac{(j-k)\pi}{N} \quad (\text{only for spin } \frac{1}{2}, \text{ Haldane - Shastry model [19 - 21]}), \quad (2.1.15)$$

and  $\mu_i$  arbitrary constants. Noting that  $\mu_i$  plays important role for the representation theory of  $Y(SU(2))$  given by Chari and Pressley ([22-24]).

The big difference between representations of Lie algebra and Yangian is in that in Yangian there appear free parameters  $\mu_i$  depending on models.

Another example for single particle is finite  $W$ -algebra ([25-26]). Denoting by  $\mathbf{L}$  and  $\mathbf{B}$  angular momentum and Lorentz boost, respectively, as well as  $D$  the dilatation operator, the set of  $\mathbf{L}$  and  $\mathbf{J}$  satisfies  $Y(SU(2))$  where ([13],[25])

$$\mathbf{I} = \mathbf{L} \quad (2.1.16)$$

$$\mathbf{J} = \mathbf{I} \times \mathbf{B} - i(D-1)\mathbf{B} \quad (2.1.17)$$

and

$$[J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma}(2\mathbf{I}^2 - c'_2 - 4)\mathbf{I}_\gamma, \quad c'_2 \text{ casimir of } SO(4, 2). \quad (2.1.18)$$

There are the following models whose Hamiltonians do commute with  $Y(SU(2))$ .

- Two component nonlinear Schrodinger equation (Murakami and Wadati [27])

$$i\psi_t = -\psi_{xx} + 2c|\psi|^2\psi, \quad (2.1.19)$$

$$\mathbf{I} = \int dx \psi_\alpha^+(x) \left(\frac{\sigma}{2}\right)_{\alpha\beta} \psi_\beta(x); \quad (2.1.20)$$

$$\mathbf{J} = -i \int dx \psi_\alpha^+(x) \left(\frac{\sigma}{2}\right)_{\alpha\beta} \psi_\beta(x) - \frac{ic}{2} \int dx dy \varepsilon(y-x) \left(\frac{\sigma}{2}\right)_{\beta\lambda} \psi_\beta^+(x) \psi_\alpha^+(y) \psi_\alpha(x) \psi_\lambda(y). \quad (2.1.21)$$

- One-dimensional Hubbard model (for  $N \rightarrow \infty$ , [28])

$$H = - \sum_{i=1}^N (a_i^+ a_{i+1} + a_{i+1}^+ a_i + b_i^+ b_{i+1} + b_{i+1}^+ b_i) - U \sum_{i=1}^N (a_i^+ a_i - \frac{1}{2})(a_i^+ a_i - \frac{1}{2}); \quad (2.1.22)$$

$$\begin{aligned} J_{\pm} &= J_1 \pm iJ_2, \\ J_+ &= \sum_{i,j} \theta_{i,j} a_i^+ b_j - U \sum_{i \neq j} \varepsilon_{i,j} I_i^+ I_j^3, \\ J_- &= \sum_{i,j} \theta_{i,j} b_i^+ a_j + U \sum_{i \neq j} \varepsilon_{i,j} I_i^- I_j^3, \\ J_3 &= \frac{1}{2} [\sum_{i,j} \theta_{i,j} (a_i^+ a_j - b_i^+ b_j) + U \sum_{i < j} \varepsilon_{i,j} I_i^+ I_j^-], \end{aligned} \quad (2.1.23)$$

where

$$\theta_{i,j} = \delta_{i,j-1} - \delta_{i,j+1}, \quad \varepsilon_{i,j} = \begin{cases} 1 & i < j, \\ 0 & i = j, \\ -1 & i > j. \end{cases} \quad (2.1.24)$$

Essler, Korepin and Schoutens found the complete solutions ([29-30]) and excitation spectrum ([31]) of 1-D Hubbard model chain.

- Haldane-Shastry model ([19-21]) whose Hamiltonian is given by a family. The first member is

$$H_2 = \sum'_{i,j} \left( \frac{Z_i Z_j}{Z_{ij} Z_{ji}} \right) (P_{ij} - 1), \quad (2.1.25)$$

where and henceforth the ' stands for  $i \neq j$  in the summation and  $P_{ij} = 2(\mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{4})$ ,  $Z_k = \exp^{i\pi \frac{k}{N}}$ ,  $Z_{ij} = Z_i - Z_j$ . The next reads

$$H_3 = \sum'_{i,j,k} \left( \frac{Z_i Z_j Z_k}{Z_{ij} Z_{jk} Z_{ki}} \right) (P_{ijk} - 1), \quad (2.1.26)$$

and

$$H_4 = \sum'_{i,j,k,l} \left( \frac{Z_i Z_j Z_k Z_l}{Z_{ij} Z_{jk} Z_{kl} Z_{li}} \right) (P_{ijkl} - 1) + H'_4, \quad (2.1.27)$$

$$H'_4 = -\frac{1}{3} H_2 - 2 \sum'_{i,j} \left( \frac{Z_i Z_j}{Z_{ij} Z_{ji}} \right)^2 (P_{ij} - 1), \quad (2.1.28)$$

where

$$\begin{aligned} P_{ijk} &= P_{ij} P_{jk} + P_{jk} P_{ki} + P_{ki} P_{ij}, \\ P_{ijkl} &= P_{ij} P_{jk} P_{kl} + (\text{cyclic for } i, j, k \text{ and } l). \end{aligned} \quad (2.1.29)$$

The eigenvalues of  $H_2$  and  $H_3$  have been solved in Ref. [21] and numerical calculations were made for  $H_4$ . The  $H_2$  and  $H_3$  were shown to be obtained in terms of quantum determinant ([32]).

- Hydrogen atom (with and without monopole, [33])

$$H = \frac{\pi^2}{2\mu} + \frac{1}{2\mu} \frac{q^2}{r^2} - \frac{\kappa}{r}, \quad \pi = p - zeA \quad (2.1.30)$$

where  $\mu$  is mass,  $q = zeg$ ,  $\kappa = ze^2$  and  $g$  being monopole charge.

- Super Yang-Mills Theory ( $N = 4$ ):  $Y(SO(6))$  ([34])

$$H = 2 \sum_{\alpha} \sum_j h(j) P_{\alpha\alpha+1}^j, \quad h(j) = \sum_{k=1}^j \frac{1}{k}, \quad h(0) = 1. \quad (2.1.31)$$

where  $P^j$  is projector for the weight  $j$  of  $SU(2)$  and  $\alpha$  stands for “lattice” index.

## 2.2 $Y(SU(3))$

For the Yangian associated to  $SU(3)$ , there are the following independent relations

$$[I_{\lambda}, I_{\mu}] = if_{\lambda\mu\nu} I_{\nu}, \quad [I_{\lambda}, J_{\mu}] = if_{\lambda\mu\nu} J_{\nu} \quad (\lambda, \mu, \nu = 1, \dots, 8). \quad (2.2.1)$$

Define

$$I_{\pm}^{(1)} = I_1 \pm iI_2, \quad U_{\pm}^{(1)} = I_6 \pm iI_7, \quad V_{\pm}^{(1)} = I_4 \mp iI_5, \quad \frac{\sqrt{3}}{2} I_8^{(1)} = I_8. \quad (2.2.2)$$

and  $J_{\mu}$  represents the corresponding operator for  $I_{\pm}^{(2)}, U_{\pm}^{(2)}, V_{\pm}^{(2)}$  and  $I_8^{(2)}, I_3^{(2)}$ . After lengthy calculation one finds that based on RTT relation there is only one independent relation for  $Y(SU(3))$  additional to equation (2.2.1):

$$[I_8^{(2)}, I_3^{(2)}] = \frac{1}{3!} (\{I_+^{(1)}, U_+^{(1)}, V_+^{(1)}\} - \{I_-^{(1)}, U_-^{(1)}, V_-^{(1)}\}) \quad (2.2.3)$$

where  $\{\dots\}$  stands for the symmetric summation. The conclusion can be verified through both the Drinfel'd formula ( $C_{\lambda\mu\nu} = if_{\lambda\mu\nu}$ ) and RTT relations with replacing  $P_{12}$  in  $SU(2)$  by

$$P_{12} = \frac{1}{3}I + \frac{1}{2} \sum_{\mu} \lambda_{\mu} \lambda_{\mu}, \quad (2.2.4)$$

where  $\lambda_{\mu}$  are the Gell-Mann matrices. Setting

$$T(u) = \sum_{n=0}^{\infty} u^{-n} T^{(n)}, \quad (2.2.5)$$

$$T^{(n)} = \begin{bmatrix} \frac{1}{3}T_0^{(n)} + T_3^{(n)} + \frac{1}{\sqrt{3}}T_8^{(n)} & T_1^{(n)} - iT_2^{(n)} & T_4^{(n)} - iT_5^{(n)} \\ T_1^{(n)} + iT_2^{(n)} & \frac{1}{3}T_0^{(n)} - T_3^{(n)} + \frac{1}{\sqrt{3}}T_8^{(n)} & T_6^{(n)} - iT_7^{(n)} \\ T_4^{(n)} + iT_5^{(n)} & T_6^{(n)} + iT_7^{(n)} & \frac{1}{3}T_0^{(n)} - \frac{2}{\sqrt{3}}T_8^{(n)} \end{bmatrix}, \quad (2.2.6)$$

and substituting them into RTT relation we find equations (2.2.1)-(2.2.3) are independent relations together with the co-product, for example,

$$\begin{aligned}\Delta I_{\pm}^{(2)} &= I_{\pm}^{(2)} \otimes 1 + 1 \otimes I_{\pm}^{(2)} \pm 2(I_3^{(1)} \otimes I_{\pm}^{(1)} - I_{\pm}^{(1)} \otimes I_3^{(1)}) \\ &+ \frac{1}{2}(V_{\mp}^{(1)} \otimes U_{\mp}^{(1)} - U_{\mp}^{(1)} \otimes V_{\mp}^{(1)})\end{aligned}\quad (2.2.7)$$

and others.

The quantum determinant of  $T(u)$  which is 3 by 3 matrix for the fundamental representation of  $gl(3)$  takes the form

$$\begin{aligned}\tilde{\det}_3 T(u) &= T_{11}(u)\{T_{22}(u-1)T_{33}(u-2) - T_{23}(u)T_{32}(u-2)\} \\ &\quad - T_{12}(u)\{T_{21}(u-1)T_{33}(u-2) - T_{23}(u-1)T_{31}(u-2)\} \\ &\quad + T_{13}(u)\{T_{21}(u-1)T_{32}(u-2) - T_{22}(u-1)T_{31}(u-2)\} \\ &= \sum_p (-1)^p T_{1p_1}(u)T_{2p_2}(u-1)T_{3p_3}(u-2)\end{aligned}\quad (2.2.8)$$

where  $p$  stands for all the possible arrangements of  $(p_1, p_2, p_3)$ . In comparison with the quantum determinant

$$\tilde{\det}_2 T(u) = \sum_{k,l,m=0}^{\infty} \frac{(l-m-1)!}{(m-1)!l!} u^{-(m+l+k)} (T_{11}^{(k)} T_{22}^{(m)} - T_{12}^{(k)} T_{21}^{(m)}), \quad (2.2.9)$$

now we have

$$\begin{aligned}\tilde{\det}_3 T(u) &= \sum_{k,l,m,p,q=0}^{\infty} \frac{(l+m-1)!}{(m-1)!l!} \frac{2^q (p+q-1)!}{(p-1)!q!} u^{-(m+l+k+p+q)} \\ &\quad \{T_{11}^{(k)} (T_{22}^{(m)} T_{33}^{(p)} - T_{23}^{(m)} T_{32}^{(p)}) - T_{12}^{(k)} (T_{21}^{(m)} T_{33}^{(p)} - T_{23}^{(m)} T_{31}^{(p)}) \\ &\quad + T_{13}^{(k)} (T_{21}^{(m)} T_{32}^{(p)} - T_{22}^{(m)} T_{31}^{(p)})\} \\ &= \sum_{n=0}^{\infty} u^{-n} C_n,\end{aligned}\quad (2.2.10)$$

i.e.,

$$C_0 = 1, C_1 = T_0^{(1)}, C_2 = T_0^{(2)} + T_0^{(1)} + 2(T_0^{(1)})^2 - \mathbf{I}^2, \quad (2.2.11)$$

$$\mathbf{I}^2 = \sum_{\lambda=1}^{\infty} \mathbf{I}_{\lambda}^2. \quad (2.2.12)$$

When we constrain  $\tilde{\det} T(u) = 1$  it leads to  $Y(SU(2))$  and  $Y(SU(3))$  that are formed by the set  $\{I_{\lambda}, J_{\lambda}\}$ ,  $\lambda = 1, 2, 3$  and  $\lambda = 1, 2, \dots, 8$  for  $SU(2)$  and  $SU(3)$ , respectively.

An example of realization of  $Y(SU(3))$  is the generalization of Haldane-Shastry model ([19-21]) for the fundamental representation of generators of  $SU(3)$ :

$$I_\mu = \sum_i F_i^\mu, \quad (2.2.13)$$

$$J_\mu = \sum_i \mu_i F_i^\mu + \lambda f_{\mu\lambda\nu} \sum_{i \neq j} W_{ij} F_i^\nu F_j^\lambda, \quad (2.2.14)$$

where  $W_{ij}$  satisfies the same relation as in Haldane-Shastry model given in section 2.1 and  $F^\mu$  are the Gell-Mann matrices.

### 2.3 $Y(SO(5))$ and $Y(SO(6))$

For  $SO(N)$  it holds

$$[L_{ij}, L_{kl}] = i C_{ij,kl}^{st} L_{st}, \quad (2.3.1)$$

where

$$C_{ij,kl}^{st} = \delta_{ik} \delta_{js} \delta_{lt} - \delta_{il} \delta_{js} \delta_{kt} - \delta_{jk} \delta_{is} \delta_{lt} + \delta_{jl} \delta_{is} \delta_{kt}. \quad (2.3.2)$$

The rational solutions of YBE for  $SO(N)$  were firstly given by Zamolodchikov's ([35]). They are also re-derived by taking the rational limit of the trigonometric R-Matrix:

$$\check{R}(u) = f(u) [u^2 P + u(A - I - \frac{3}{2}P)\xi + \frac{3}{2}I\xi^2], \quad (2.3.3)$$

where  $u$  stands for spectral parameter and  $\xi$  the other free parameter ([36-37]). The elements of  $\check{R}(u)$  are  $(a, b, c, d = -2, -1, 0, 1, 2)$

$$[\check{R}(u)]_{cd}^{ab} = u^2 \delta_{ab} \delta_{bc} + u(\delta_{a-b} \delta_{c-d} - \delta_{ac} \delta_{bd} - \frac{3}{2} \delta_{ad} \delta_{bc}) \xi + \frac{3}{2} \delta_{ac} \delta_{bd} \xi^2. \quad (2.3.4)$$

For  $SO(5)$ , we introduce

$$T^{(1)} = \xi \begin{bmatrix} E_3 - \frac{3}{2} & U_+ & E_+ & V_+ & 0 \\ U_- & F_3 - \frac{3}{2} & F_+ & 0 & -V_+ \\ E_- & F_- & -\frac{3}{2} & -F_+ & -E_+ \\ V_- & 0 & -F_- & -F_3 - \frac{3}{2} & -U_+ \\ 0 & -V_- & -E_- & -U_- & -E_3 - \frac{3}{2} \end{bmatrix}, \quad (2.3.5)$$

where

$$\begin{aligned} E_3 &= E_{22} - E_{-2,-2}, & F_3 &= E_{11} - E_{-1,-1}, & U_+ &= E_{21} - E_{-1,-2}, \\ V_+ &= E_{2,-1} - E_{1,-2}, & E_+ &= E_{20} - E_{0,-2}, & F_+ &= E_{10} - E_{0,-1}, \\ U_- &= E_{12} - E_{-2,-1}, & V_- &= E_{-12} - E_{-2} & E_- &= E_{02} - E_{-20}, \\ F_- &= E_{01} - E_{-10}; \end{aligned} \quad (2.3.6)$$



$$T_{ab}^{(2)} = \frac{3}{2}\xi^2 E_{ab}^{(2)} \quad (a, b = -2, -1, 0, 1, 2). \quad (2.3.7)$$

Substituting  $T^{(n)}$  (only  $n = 1, 2$  are needed to be considered) into RTT relation, there appears 35 relations for  $J_\mu$  besides the Jacobi identities. However, a lengthy computation shows that besides

$$\begin{aligned} [I_\alpha, I_\beta] &= C_{\alpha\beta}^\gamma I_\gamma \\ [I_\alpha, J_\beta] &= C_{\alpha\beta}^\gamma J_\gamma \end{aligned} \quad (\alpha = i, j), \quad (2.3.8)$$

there is only one independent relation

$$[E_3^{(2)}, F_3^{(2)}] = \frac{1}{4!}(\{U_-, E_+, F_-\} - \{U_+, E_-, F_+\} - \{V_+, E_-, F_-\} + \{V_-, E_+, F_+\}), \quad (2.3.9)$$

where again  $\{ \}$  stands for the symmetric summation.

A realization of  $Y(SO(5))$  is given as follows. Set

$$I_{ab}(x) = \frac{1}{2}\psi_\alpha^+(x)(I^{ab})_{\alpha\beta}\psi_\beta(x) \quad (a, b = -2, -1, 0, 1, 2), \quad (2.3.10)$$

$$\{\psi_\alpha^+(x), \psi_\beta(y)\}_+ = \delta(x-y)\delta_{\alpha\beta}. \quad (2.3.11)$$

Then

$$I_{ab} = \sum_x I_{ab}(x), \quad (2.3.12)$$

$$J_{ab} = \sum_{x,y,c \neq a,b} \epsilon(x-y)I_{ac}(x)I_{cb}(y) \quad (2.3.13)$$

satisfies the commuting relations for  $Y(SO(5))$ . The following Hamiltonian of ladder model not only commutes with  $I_{ab}$ , i.e., it possesses  $SO(5)$  symmetry, but also commutes with  $J_{ab}$ .

$$H = H_1 + \sum_x H_2(x) + \sum_x H_3(x); \quad (2.3.14)$$

$$H_1 = 2t_1 \sum_{\langle x,y \rangle} [c_\sigma^+(x)c_\sigma(y) + d_\sigma^+(x)d_\sigma(y) + H.C.]; \quad (2.3.15)$$

$$\begin{aligned} H_2(x) &= U(n_{c\uparrow} - \frac{1}{2})(n_{c\downarrow} - \frac{1}{2}) + (c \rightarrow d) + V(n_c - 1)(n_d - 1) + J\mathbf{S}_c \cdot \mathbf{S}_d \\ &= \frac{J}{4} \sum_{a < b} I_{ab}^2 + (\frac{1}{8}J + \frac{1}{2}U)(\psi_\alpha^+ \psi_\alpha - 2); \end{aligned} \quad (2.3.16)$$

$$H_3(x) = -2t_3(c_\sigma^+(x)d_\sigma(x) + H.C.). \quad (2.3.17)$$

Because locally  $SO(6) \simeq SU(4)$  we introduce (15 generators)

$$T_{ab}^{(1)} = I_{ab}, \quad T_{ab}^{(2)} = I_{ab}^{(2)}(a, b = 1, 2, \dots, 6). \quad (2.3.18)$$

and the  $\check{R}(u)$ -matrix reads

$$\check{R}(u) = f(u)[u^2 P + u\xi(A - 2P - I) + 2\xi^2 I]. \quad (2.3.19)$$

The RTT relation gives  $4 + 4 + 441 + 315 + 225$  more relations. After careful calculations one finds ([15-16]) that there are the following independent relations for  $J_{ab}$  themselves:

$$\begin{aligned} [I_{12}^{(2)}, I_{34}^{(2)}] &= \frac{i}{24}(\{I_{23}, I_{16}, I_{46}\} + \{I_{23}, I_{15}, I_{45}\} + \{I_{14}, I_{25}, I_{35}\} \\ &\quad + \{I_{14}, I_{26}, I_{36}\} - \{I_{13}, I_{26}, I_{46}\} - \{I_{13}, I_{25}, I_{45}\} \\ &\quad - \{I_{24}, I_{15}, I_{35}\} - \{I_{24}, I_{16}, I_{36}\}); \end{aligned} \quad (2.3.20)$$

$$\begin{aligned} [I_{12}^{(2)}, I_{56}^{(2)}] &= \frac{i}{24}(\{I_{15}, I_{23}, I_{36}\} + \{I_{15}, I_{24}, I_{46}\} + \{I_{26}, I_{13}, I_{35}\} \\ &\quad + \{I_{26}, I_{14}, I_{45}\} - \{I_{25}, I_{13}, I_{36}\} - \{I_{25}, I_{14}, I_{46}\} \\ &\quad - \{I_{16}, I_{23}, I_{35}\} - \{I_{16}, I_{24}, I_{45}\}); \end{aligned} \quad (2.3.21)$$

$$\begin{aligned} [I_{34}^{(2)}, I_{56}^{(2)}] &= \frac{i}{24}(\{I_{45}^{(1)}, I_{13}^{(1)}, I_{16}^{(1)}\} + \{I_{45}^{(1)}, I_{23}^{(1)}, I_{26}^{(1)}\} + \{I_{36}^{(1)}, I_{14}^{(1)}, I_{16}^{(1)}\} \\ &\quad + \{I_{36}^{(1)}, I_{24}^{(1)}, I_{26}^{(1)}\} - \{I_{35}^{(1)}, I_{14}^{(1)}, I_{16}^{(1)}\} - \{I_{35}^{(1)}, I_{24}^{(1)}, I_{26}^{(1)}\} \\ &\quad - \{I_{46}^{(1)}, I_{13}^{(1)}, I_{16}^{(1)}\} - \{I_{46}^{(1)}, I_{23}^{(1)}, I_{26}^{(1)}\}). \end{aligned} \quad (2.3.22)$$

### 3 Applications of Yangian

The first example was given by Belavin ([38]) in deriving the spectrum of nonlinear  $\sigma$  model. Here we only show briefly some interpretations of Yangian through the particular realizations of Yangian.

#### 3.1 Reduction of $Y(SU(2))$

The simplest realization of  $Y(SU(2))$  is made of two-spin system with  $\mathbf{S}_1$  and  $\mathbf{S}_2$  (any dimensional representations of  $SU(2)$ ):

$$\mathbf{J}' = \frac{\mathbf{1}}{\mu + \nu} \mathbf{J} = \frac{\mathbf{1}}{\mu + \nu} (\mu \mathbf{S}_1 \times \mathbf{1} + \nu \mathbf{S}_2 \times \mathbf{1} + 2\lambda \mathbf{S}_1 \times \mathbf{S}_2), \quad (3.1.1)$$

that contains the (antisymmetric) tensor interaction between  $\mathbf{S}_1$  and  $\mathbf{S}_2$ . For example, for Hydrogen atom  $\mathbf{S}_1 = \mathbf{L}$  and  $\mathbf{S}_2 = \mathbf{K}$  (Lung-Lenz vector).

For  $S_1 = S_2 = 1/2$ , when

$$\mu\nu = \lambda^2, \quad (3.1.2)$$

we prove that after the following similar transformation

$$\mathbf{Y} = A\mathbf{J}'A^{-1}, \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \nu & i\lambda & 0 \\ 0 & i\lambda & \nu & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (3.1.3)$$

the Yangian reduces to  $SO(4)$ : ( $\rho = \nu + i\lambda = \sqrt{\nu^2 + \lambda^2}e^{i\theta}$ )

$$\begin{aligned} Y_1 &= \begin{bmatrix} M_1 & 0 \\ 0 & L_1 \end{bmatrix}, \quad M_1 = \frac{1}{2} \begin{bmatrix} 0 & \rho \\ \rho^{-1} & 0 \end{bmatrix}, \quad L_1 = \frac{1}{2} \begin{bmatrix} 0 & \rho^{-1} \\ \rho & 0 \end{bmatrix}, \\ Y_2 &= \begin{bmatrix} M_2 & 0 \\ 0 & L_2 \end{bmatrix}, \quad M_2 = \frac{1}{2} \begin{bmatrix} 0 & -i\rho \\ i\rho^{-1} & 0 \end{bmatrix}, \quad L_2 = \frac{1}{2} \begin{bmatrix} 0 & -i\rho^{-1} \\ i\rho & 0 \end{bmatrix}, \\ Y_3 &= \begin{bmatrix} \frac{1}{2}\sigma_3 & 0 \\ 0 & \frac{1}{2}\sigma_3 \end{bmatrix}, \quad M_3 = \frac{1}{2}\sigma_3. \end{aligned} \quad (3.1.4)$$

and

$$\mathbf{Y}^2 = \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4}. \quad (3.1.5)$$

Namely, under  $\mu\nu = \lambda^2$ , the  $\mathbf{Y}$  reduces to  $SO(4)$  by  $M_{\pm} = M_1 \pm iM_2$ ,  $M_+ = \rho\sigma_+$ ,  $M_- = \rho^{-1}\sigma_-$ . The scaled  $M_{\pm}$  and  $M_3$  still satisfy the  $SU(2)$  relations:

$$[M_3, M_{\pm}] = \pm M_{\pm}, \quad [M_+, M_-] = 2M_3. \quad (3.1.6)$$

and there are the similar relations for  $\mathbf{L}$ .

It should be emphasized that here the new “spin”  $\mathbf{M}$  (and  $\mathbf{L}$ ) is the consequence of two  $\text{spin}(\frac{1}{2})$  interaction. As usual for two 2-dimensional representations of  $SU(2)$  (Lie algebra)

$$\underline{2} \otimes \underline{2} = \underline{3} \text{ (spin triplet)} \oplus \underline{1} \text{ (singlet)}. \quad (3.1.7)$$

However, here we meet a different decomposition:

$$\underline{2} \otimes \underline{2} = \underline{2}(\mathbf{M}) \oplus \underline{2}(\mathbf{L}). \quad (3.1.8)$$

The idea can be generalized to  $SU(3)$ 's fundamental representation

$$J_{\lambda} = uI_1^{\lambda} + vI_2^{\lambda} + \lambda f_{\lambda\mu\nu} \sum_{i < j} F_{1i}^{\mu} F_{2j}^{\nu}, \quad (3.1.9)$$

$$[F_{i\mu}, F_{j\nu}] = if_{\mu\nu\lambda} F_{i\lambda} \delta_{ij} \quad (\lambda, \mu, \nu = 1, 2, \dots, 8). \quad (3.1.10)$$

Under the condition

$$uv = \lambda^2, \quad v + i\lambda = \rho, \quad (3.1.11)$$

and the similar transformation

$$Y_\mu = AJ_\mu A^{-1}/(u+v), \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & i\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 & 0 & 0 & i\lambda & 0 & 0 \\ 0 & i\lambda & 0 & \nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu & 0 & i\lambda & 0 \\ 0 & 0 & i\lambda & 0 & 0 & 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\lambda & 0 & \nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (3.1.12)$$

the Yangian then reduces to

$$\begin{aligned} Y(I_-) &= \begin{bmatrix} \rho^{-1}I_- & 0 & 0 \\ 0 & \rho I_- & 0 \\ 0 & 0 & I_- \end{bmatrix}, \quad Y(I_+) = \begin{bmatrix} \rho I_+ & 0 & 0 \\ 0 & \rho^{-1}I_- & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \\ Y(I_8) &= \frac{\sqrt{3}}{3} \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad Y(I_3) = \frac{1}{2} \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \\ Y(U_+) &= \begin{bmatrix} U_+ & 0 & 0 \\ 0 & \rho U_+ & 0 \\ 0 & 0 & \rho^{-1}U_+ \end{bmatrix}, \quad Y(U_-) = \begin{bmatrix} U_- & 0 & 0 \\ 0 & \rho^{-1}U_- & 0 \\ 0 & 0 & \rho U_- \end{bmatrix}, \\ Y(V_+) &= \begin{bmatrix} \rho^{-1}V_- & 0 & 0 \\ 0 & V_- & 0 \\ 0 & 0 & \rho V_- \end{bmatrix}, \quad Y(V_-) = \begin{bmatrix} \rho V_- & 0 & 0 \\ 0 & V_- & 0 \\ 0 & 0 & \rho^{-1}V_- \end{bmatrix}. \end{aligned} \quad (3.1.13)$$

The usual decomposition through the Clebsch-Gordan coefficients for the representations of Lie algebra  $SU(3)$  is  $\underline{3} \otimes \underline{3} = \underline{6} \oplus \underline{3}$ . However, here we have

$$\underline{3} \otimes \underline{3} = \underline{3} \oplus \underline{3} \oplus \underline{3}, \quad (3.1.14)$$

and

$$\sum_{\lambda=1}^8 Y_\lambda^2 = \frac{1}{u+v} \sum_{\lambda=1}^8 J_\lambda^2 = \frac{1}{3}. \quad (3.1.15)$$

It is easy to check that the rescaling factor  $\rho$  does not change the commutation relations for  $SU(3)$  formed by  $I_\pm$ ,  $U_\pm$ ,  $V_\pm$ ,  $I_3$  and  $I_8$ . In general, we guess for the fundamental representation of  $SU(n)$  we shall meet

$$\underline{n} \otimes \underline{n} = \underline{n} \oplus \underline{n} \oplus \underline{n} + \cdots + \underline{n} \quad (n \text{ times}). \quad (3.1.16)$$

Next we consider Yang-Mills gauge field for reduced  $Y(SU(2))$ . For a tensor wave function ( $x \equiv \{x_1, x_2, x_3, x_0\}$ ),

$$\Psi(x) = \|\psi_{ij}(x)\| \quad (i, j = 1, 2, 3, 4). \quad (3.1.17)$$

An isospin transformation yields

$$\Psi'(x) = U(x)\Psi(x), \quad U(x) = 1 - i\theta^a J_a, \quad (3.1.18)$$

where

$$J^a = uS_a \otimes \mathbf{1} + v\mathbf{1} \otimes S_a + 2\lambda\epsilon_{abc}S^b \otimes S^c, \quad (3.1.19)$$

or

$$[J_a]_{\gamma\delta}^{\alpha\beta} = u(S^a)_{\alpha\gamma}\delta_{\beta\delta} + v(S^a)_{\beta\delta}\delta_{\alpha\gamma} + i\alpha\epsilon_{abc}(S^b)_{\alpha\gamma}(S^c)_{\beta\delta}. \quad (3.1.20)$$

Define

$$D_\mu = \partial_\mu + gA_\mu, \quad (3.1.21)$$

i.e.,

$$[D_\mu\psi]_{\alpha\beta} = \partial_\mu\psi_{\alpha\beta} + gA_\mu^a[Y_a]_{\gamma\delta}^{\alpha\beta}\psi_{\gamma\delta}(x), \quad A_\mu = A_\mu^a J_a. \quad (3.1.22)$$

The gauge-covariant derivative should preserve

$$\delta(D_\mu\psi) = 0, \quad (3.1.23)$$

i.e.,

$$(-i\partial_\mu\theta^a(x) + g\delta A_\mu^a)[Y_a]_{\gamma\delta}^{\alpha\beta} - ig\theta^a(x)A_\mu^b[J_b, J_a]_{\gamma\delta}^{\alpha\beta} = 0. \quad (3.1.24)$$

When  $uv = \lambda^2$  and by rescaling

$$Y_a = (u + v)J_a, \quad (3.1.25)$$

we have

$$\delta A_\mu^a = \epsilon_{abc}\theta^b(x)A_\mu^c(x) + \frac{i}{g}\partial_\mu\theta^a(x), \quad (3.1.26)$$

and

$$F_{\mu\nu} = \frac{1}{g}[D_\mu, D_\nu] = F_{\mu\nu}^a Y_a, \quad (3.1.27)$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ig\epsilon_{abc}A_\mu^b A_\nu^c. \quad (3.1.28)$$

Here the tensor isospace has been separated to two irrelevant spaces, i.e.,  $\Psi = \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix}$  where  $\Psi_1$  and  $\Psi_2$  are  $2 \times 2$  wavefunction.

### 3.2 Illustrative examples: NMR of Breit-Rabi Hamiltonian and Yangian

The Breit-Rabi Hamiltonian is given by

$$H = \mathbf{K} \cdot \mathbf{S} + \mu \mathbf{B} \cdot \mathbf{S}, \quad (3.2.1)$$

where  $S = \frac{1}{2}$  and  $B = \mathbf{B}(t)$  is magnetic field.

The Hamiltonian can easily be diagonalized for any background angular momentum (or spin)  $\mathbf{K}$ . The  $\mathbf{S}$  stands for spin of electron and for simplicity  $\mathbf{K} = \mathbf{S}_1$  ( $S_1 = 1/2$ ) is an average background spin contributed by other source, say, control spin. Denoting by

$$H = H_0 + H_1(t), \quad H_0 = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2, \quad H_1(t) = \mu \mathbf{B}(t) \cdot \mathbf{S}_2. \quad (3.2.2)$$

Let us work in the interaction picture:

$$H_I = \mu \mathbf{B}(t) \cdot (e^{i\alpha \mathbf{S}_1 \cdot \mathbf{S}_2} \mathbf{S}_2 e^{-i\alpha \mathbf{S}_1 \cdot \mathbf{S}_2}) = \mu \mathbf{B}(t) \cdot \mathbf{J}, \quad (3.2.3)$$

$$\mathbf{J} = \mu_1 \mathbf{S}_1 + \mu_2 \mathbf{S}_2 + 2\lambda (\mathbf{S}_1 \times \mathbf{S}_2), \quad (3.2.4)$$

where  $\mu_1 = \frac{1}{2}(1 - \cos\alpha)$ ,  $\mu_2 = \frac{1}{2}(1 + \cos\alpha)$ ,  $\lambda = \frac{1}{2}\sin\alpha$ . Obviously, here we have  $\mu_1\mu_2 = \lambda^2$ . It is not surprising that the  $Y(SU(2))$  reduces to  $SO(4)$  here because the transformation is fully Lie-algebraic operation. This is an exercise in quantum mechanics.

For generalization we regard  $\mu_1$  and  $\mu_2$  as independent parameters, i.e., drop the relation  $\mu_1\mu_2 = \lambda^2$ . Looking at

$$\mathbf{J} = \mu_1 \mathbf{S}_1 + \mu_2 \mathbf{S}_2 - \frac{1}{2}(\mu_1 + \mu_2)(\mathbf{S}_1 + \mathbf{S}_2) + \gamma(\mathbf{S}_1 + \mathbf{S}_2) + 2\lambda \mathbf{S}_1 \times \mathbf{S}_2. \quad (3.2.5)$$

When  $\gamma = \frac{1}{2}$ ,  $\mu_2 - \mu_1 = \cos\alpha$  and  $\lambda = \frac{1}{2}\sin\alpha$ , it reduces to the form in the interacting picture. Putting

$$\mathbf{S}_1 + \mathbf{S}_2 = \mathbf{S}, \quad 2\lambda = -\frac{h}{2} (h \text{ is not Plank constant}). \quad (3.2.6)$$

In accordance with the convention we have

$$\mathbf{J} = \gamma \mathbf{S} + \sum_{i=1}^2 \mu_i \mathbf{S}_i + \frac{h}{2} \mathbf{S}_1 \times \mathbf{S}_2 - \frac{1}{2}(\mu_1 + \mu_2) \mathbf{S} = \gamma \mathbf{S} + \mathbf{Y}. \quad (3.2.7)$$

Since  $\mathbf{J} \rightarrow \xi \mathbf{S} + \mathbf{J}$  still satisfies Yangian relations, it is natural to appear the term  $\gamma \mathbf{S}$ . The interacting Hamiltonian then reads

$$H_I(t) = -\gamma \mathbf{B}(t) \cdot \mathbf{S} - \mathbf{B}(t) \cdot \mathbf{Y}. \quad (3.2.8)$$

When  $\mu_i = 0$ ,  $h = 0$ , it is the usual NMR for spin  $1/2$ . To solve the equation, we use

$$i\frac{\partial\Psi(t)}{\partial t} = H_I(t)\Psi(t), \quad |\Psi(t)\rangle = \sum_{\alpha=\pm,3;0} a_\alpha(t)|\chi_\alpha\rangle, \quad (3.2.9)$$

where  $\{\chi_\pm, \chi_3\}$  is the spin triplet and  $\chi_0$  singlet. Setting

$$B_\pm(t) = B_1(t) \pm iB_2(t) = B_1 e^{\mp i\omega_0 t}, \quad \text{and} \quad B_3 = \text{const.} \quad (3.2.10)$$

and rescaling by

$$a_\pm(t) = e^{\pm i\omega_0 t} b_\pm(t), \quad (3.2.11)$$

we get

$$\begin{aligned} i\frac{db_\pm(t)}{dt} &= -\gamma\left\{\frac{1}{\sqrt{2}}B_1a_3(t) \mp (\omega_0\gamma^{-1} - B_3)b_\pm(t)\right\} \pm \frac{1}{2\sqrt{2}}\mu_-B_1a_0(t), \\ i\frac{da_3(t)}{dt} &= -\frac{\gamma B_1}{\sqrt{2}}\{b_+(t) + b_-(t)\} - \frac{1}{2}\mu_-B_3a_0(t), \\ i\frac{da_0(t)}{dt} &= -\frac{1}{2}\mu_+\left\{\frac{1}{\sqrt{2}}B_1[b_-(t) - b_+(t)]\right\} + B_3a_3(t), \end{aligned} \quad (3.2.12)$$

where  $\mu_\pm = (\mu_1 - \mu_2 \pm i\frac{h}{2})$ , i.e.,

$$|\Phi(t)\rangle = \begin{bmatrix} b_1(t) \\ a_3(t) \\ b_-(t) \\ a_0(t) \end{bmatrix}, \quad \mathcal{H}_I = \begin{bmatrix} \omega_0 - \gamma B_3 & -\gamma B_1 \frac{1}{\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}}\mu_-B_1 \\ -\gamma B_1 \frac{1}{\sqrt{2}} & 0 & -\gamma B_1 \frac{1}{\sqrt{2}} & -\frac{1}{2}\mu_-B_3 \\ 0 & -\gamma B_1 \frac{1}{\sqrt{2}} & -(\omega_0 - \gamma B_3) & -\frac{1}{2\sqrt{2}}\mu_-B_1 \\ \frac{1}{2\sqrt{2}}\mu_+B_1 & -\frac{1}{2}\mu_+B_3 & -\frac{1}{2\sqrt{2}}\mu_+B_1 & 0 \end{bmatrix}, \quad (3.2.13)$$

$$i\frac{d|\Phi(t)\rangle}{dt} = H_I|\Phi(t)\rangle. \quad (3.2.14)$$

Noting that  $\mathcal{H}_I$  is independent of time, we get

$$|\Phi(t)\rangle = e^{-iEt}|\Phi(0)\rangle. \quad (3.2.15)$$

Then

$$\det|H_I - E| = 0 \quad (3.2.16)$$

leads to

$$\begin{aligned} E^4 - [(\omega_1 - \gamma B_3)^2 + \gamma^2 B_1^2 + \frac{1}{4}\mu_+\mu_-(B_1^2 + B_3^2)]E^2 + \\ \frac{1}{4}\mu_+\mu_-[B_3^2(\omega_0 - \gamma B_3)^2 - 2\gamma B_3 B_1^2(\omega_0 - \gamma B_3) + \gamma^2 B_1^4] = 0. \end{aligned} \quad (3.2.17)$$

There is a transition between the spin singlet and triplet in the NMR process, i.e., the Yangian transfers the quantum information through the evolution. The simplest case is  $B_1 = 0$ , then the eigenvalues are

$$E = \pm(\omega_0 - \gamma B_3), E = \pm\omega = \pm \frac{B_3}{2} \sqrt{(\mu_1 - \mu_2)^2 + \frac{\hbar^2}{4}}. \quad (3.2.18)$$

It turns out that there is a vibration between  $s = 0$  and  $s = 1$ .

$$\langle s^2 \rangle = 0 \text{ at } t = \frac{\pi}{2\omega} \text{ (total spin } = 0), \quad (3.2.19)$$

$$\langle s^2 \rangle = 2 \text{ at } t = \frac{\pi}{\omega} \text{ (total spin } = 1). \quad (3.2.20)$$

Under adiabatic approximation it can be proved that it appears Berry's phase. Obviously, only spin vector can make the stereo angle. The role of spin singlet here is a witness that shares energy of spin=1 state.

Actually, if

$$B_{\pm}(t) = B_0 \sin \theta e^{\mp i\omega_0 t}, \quad B_3 = B_0 \cos \theta, \quad (3.2.21)$$

and

$$\begin{aligned} |\chi_{11}\rangle &= |\uparrow\uparrow\rangle, \quad |\chi_{1-1}\rangle = |\downarrow\downarrow\rangle, \quad |\chi_{10}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \\ |\chi_{00}\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \end{aligned} \quad (3.2.22)$$

then let us consider the eigenvalues of

$$H = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2 - \gamma B_0 S_3 - g B_0 J_3, \quad (3.2.23)$$

under adiabatic approximation which are

$$E_{\pm} = \frac{1}{2} \left( -\frac{\alpha}{2} \pm \sqrt{\alpha^2 + g^2 B_0^2 \mu_+ \mu_-} \right), \quad (3.2.24)$$

and

$$f_1^{(\pm)} = [2(\alpha^2 + g^2 B_0^2 \mu_+ \mu_-)]^{-1/2} [(\alpha^2 + g^2 B_0^2 \mu_+ \mu_-)^{1/2} \pm \alpha]^{1/2}, \quad (3.2.25)$$

$$f_2^{(\pm)} = [2(\alpha^2 + g^2 B_0^2 \mu_+ \mu_-)]^{-1/2} \left[ \frac{\mu_+}{\mu_-} (\alpha^2 + g^2 B_0^2 \mu_+ \mu_-)^{1/2} \mp \alpha \right]^{1/2}. \quad (3.2.26)$$

We obtain the eigenstates of  $H$  besides  $|\chi_{1i}\rangle$  ( $i = 1, 2$ )

$$|\chi_{\pm}\rangle = f_1^{(\pm)} |\chi_{10}\rangle + f_2^{(\pm)} |\chi_{00}\rangle, \quad (3.2.27)$$



where

$$\begin{aligned}
|\chi_{11}(t)\rangle &= \cos^2 \frac{\theta}{2} |\chi_{11}\rangle + \frac{1}{\sqrt{2}} \sin \theta e^{-i\omega_0 t} |\chi_{10}\rangle + \sin^2 \frac{\theta}{2} e^{-2i\omega_0 t} |\chi_{1-1}\rangle, \\
|\chi_{1-1}(t)\rangle &= \sin^2 \frac{\theta}{2} e^{2i\omega_0 t} |\chi_{11}\rangle - \frac{1}{\sqrt{2}} \sin \theta e^{i\omega_0 t} |\chi_{10}\rangle + \cos^2 \frac{\theta}{2} |\chi_{1-1}\rangle, \\
|\chi_{\pm}(t)\rangle &= \frac{1}{\sqrt{2}} f_1^{(\pm)} \{ -\sin \theta e^{i\omega_0 t} |\chi_{11}\rangle + \sqrt{2} \cos \theta |\chi_{10}\rangle + \sin \theta e^{-i\omega_0 t} |\chi_{1-1}\rangle \} \\
&\quad + f_2^{(\pm)} |\chi_{00}\rangle.
\end{aligned} \tag{3.2.28}$$

We then obtain

$$\begin{aligned}
\langle \chi_{11}(t) | \frac{\partial}{\partial t} | \chi_{11}(t) \rangle &= -i\omega_0(1 - \cos \theta), \\
\langle \chi_{1-1}(t) | \frac{\partial}{\partial t} | \chi_{11}(t) \rangle &= i\omega_0(1 - \cos \theta), \\
\langle \chi_{\pm}(t) | \frac{\partial}{\partial t} | \chi_{\pm}(t) \rangle &= 0.
\end{aligned} \tag{3.2.29}$$

The Berry's phase is then

$$\gamma_{1\pm 1} = \pm \Omega, \quad \Omega = 2\pi(1 - \cos \theta), \tag{3.2.30}$$

whereas  $\gamma_{10} = \gamma_{00} = 0$ . The Yangian changes the eigenstates of  $H$ , but preserves the Berry's phase.

### 3.3 Transition between S-wave and P-wave superconductivity

We set for a pair of electrons:

$$S : \quad \text{spin singlet, } L = 0; \tag{3.3.1}$$

$$P : \quad \text{spin triplet, } L = 1. \tag{3.3.2}$$

Due to Balian-Werthamer ([39]), we have

$$\Delta(\mathbf{k}) = -\frac{1}{2} \sum_{\mathbf{k}'} V(\mathbf{k}, \mathbf{k}') \frac{\Delta(\mathbf{k}')}{E(\mathbf{k}')} \tanh \frac{\beta}{2} E(\mathbf{k}'), \tag{3.3.3}$$

$$E(\mathbf{k}) = (\epsilon^2(k) + |\Delta(\mathbf{k})|^2)^{\frac{1}{2}}. \tag{3.3.4}$$

Therefore, still by Balian-Werthamer ([39]), we know

$$\Delta(\mathbf{k}) = \Delta(k) \left( \frac{4\pi}{3} \right)^{\frac{1}{2}} \begin{bmatrix} \sqrt{2} Y_{1,1}(\hat{\mathbf{k}}) & Y_{1,0}(\hat{\mathbf{k}}) \\ Y_{1,0}(\hat{\mathbf{k}}) & \sqrt{2} Y_{1,-1}(\hat{\mathbf{k}}) \end{bmatrix}^* = (-\sqrt{6}) \Delta(k) \left( \frac{4\pi}{3} \right)^{\frac{1}{2}} \Phi_{0,0}(\hat{\mathbf{k}}), \tag{3.3.5}$$

$$\Phi_{0,0}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{3}}\{Y_{1,-1}(\hat{\mathbf{k}})\chi_{11} - Y_{1,0}(\hat{\mathbf{k}})\chi_{10} + Y_{1,1}(\hat{\mathbf{k}})\chi_{1-1}\} = \frac{1}{\sqrt{8}} \begin{bmatrix} \hat{\mathbf{k}}_- & -\hat{\mathbf{k}}_z \\ -\hat{\mathbf{k}}_z & -\hat{\mathbf{k}}_+ \end{bmatrix}, \quad (3.3.6)$$

where  $\chi_{11}, \chi_{10}$  and  $\chi_{1-1}$  stand for spin triplet:

$$\Phi_{0,0} \equiv \Phi_{J=0,m=0}. \quad (3.3.7)$$

The wave function of SC is

$$\phi_{0,0} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & Y_{0,0} \\ -Y_{0,0} & 0 \end{bmatrix}. \quad (3.3.8)$$

Introducing

$$I_\mu = \sum_{i=1}^2 S_\mu(i); \quad (\mu = 1, 2, 3), \quad (3.3.9)$$

$$J_\mu = \sum_{i=1}^2 \lambda_i S_\mu(i) - \frac{ihv}{4} \epsilon_{\mu\lambda\nu} (S^\lambda(1)S^\nu(2) - S^\lambda(2)S^\nu(1)), \quad (3.3.10)$$

and noting that  $J_\mu \rightarrow J_\mu + fI_\mu$  does not change the Yangian relations, we choose for simplicity  $f = -\frac{1}{2}(\lambda_1 + \lambda_2)$ . Then we obtain for  $G = \hat{\mathbf{k}} \cdot (\mathbf{J} + f\mathbf{I})$

$$G\phi_{0,0} = \hat{\mathbf{k}} \cdot (\mathbf{J} + f\mathbf{I})\phi_{0,0} = \frac{\sqrt{3}}{2}(\lambda_2 - \lambda_1 + \frac{hv}{2})\Phi_{0,0}, \quad (3.3.11)$$

$$G\Phi_{0,0} = \hat{\mathbf{k}} \cdot (\mathbf{J} + f\mathbf{I})\Phi_{0,0} = \frac{1}{2\sqrt{3}}(\lambda_2 - \lambda_1 - \frac{hv}{2})\phi_{0,0}. \quad (3.3.12)$$

The transition directionally depends on the parameters in  $Y(SU(2))$ . For instance,

$$SC \rightarrow PC : G\phi_{0,0} = \frac{\sqrt{3}}{2}\Phi_{0,0}, \quad G\Phi_{0,0} = 0, \quad \text{if } \lambda_1 - \lambda_2 = -\frac{hv}{2}, \quad (3.3.13)$$

and

$$PC \rightarrow SC : G\phi_{0,0} = 0, \quad G\Phi_{0,0} = -\frac{hv}{2\sqrt{3}}\phi_{0,0}, \quad \text{if } \lambda_1 - \lambda_2 = \frac{hv}{2}. \quad (3.3.14)$$

We call the type of the transition “directional transition” ([40]). The controlled parameters are in the Yangian operation. They represent the interaction coming from other controlled spin.

We have got used to apply electromagnetic field  $A_\mu$  to make transitions between  $l$  and  $l \pm 1$  states. Now there is Yangian formed by two spins that plays the role changing angular momentum states.

### 3.4 $Y(SU(3))$ -directional transitions

Setting

$$F_\mu = \frac{1}{2}\lambda_\mu, \quad [F_\lambda, F_\mu] = if_{\lambda\mu\nu}F_\nu, \quad (3.4.1)$$

$$I_\mu = \sum_i F_i^\nu, \quad (3.4.2)$$

$$J_\mu = \sum_i \mu_i F_i^\mu - if_{\mu\nu\lambda} \sum_{i \neq j} W_{ij} F_i^\nu F_j^\lambda, \quad (W_{ij} = -W_{ji}), \quad (3.4.3)$$

$$[F_i^\lambda, F_j^\mu] = if_{\lambda\mu\nu} \delta_{ij} F_i^\nu, \quad (3.4.4)$$

where  $\{F_\mu\}$  is the fundamental representation of  $SU(3)$  and  $(i, j, k = 1, 2, \dots, 8)$

$$\triangle_{ijk} = W_{ij}W_{jk} + W_{jk}W_{ki} + W_{ki}W_{ij} = -1. \quad (3.4.5)$$

(Here, no summation over repeated indices,  $i \neq j \neq k$ ). The reason that such a condition works only for 3-dimensional representation of  $SU(3)$  is similar to Haldane's (long-ranged) realization of  $Y(SU(2))$  ([19]). In  $SU(2)$  long-ranged form, the property of Pauli matrices leads to  $(\sigma^\pm)^2 = 0$ . Instead, for  $SU(3)$  the conditions of  $J_\mu$  satisfying  $Y(SU(3))$  read

$$\sum_{i \neq j} (1 - w_{ij}^2) (I_j^+ V_i^+ U_i^+ - U_i^- V_i^- I_j^- + I_i^+ V_j^+ U_i^+ - U_i^- V_j^- I_i^- + I_j^+ V_j^+ U_i^+ - U_i^- V_j^- I_j^-) = 0, \quad (3.4.6)$$

and

$$\sum_i (I_i^+ V_i^+ U_i^+ - U_i^- V_i^- I_i^-) = 0, \quad (3.4.7)$$

that are satisfied for Gell-Mann matrices.

The simplest realization of  $Y(SU(3))$  is then

$$W_{ij} = \begin{cases} 1 & i > j \\ 0 & i = j \\ -1 & i < j \end{cases} \quad (W_{ij} = -W_{ji}). \quad (3.4.8)$$

Recalling  $(I_8 = \frac{\sqrt{3}}{2}Y)$

$$\begin{aligned} I^+ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad V^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ I^3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned} \quad (3.4.9)$$

We find

$$\begin{aligned}
J_\mu &= \{\bar{I}_\pm, \bar{U}_\pm, \bar{V}_\pm, \bar{I}_3, \bar{I}_8\}, \\
\bar{I}_\pm &= \sum_i \mu_i I_i^\pm \mp 2h \sum_{i \neq j} W_{ij} (I_i^\pm I_j^3 + \frac{1}{2} U_i^\mp V_j^\mp), \\
\bar{U}_\pm &= \sum_i \mu_i U_i^\pm \pm h \sum_{i \neq j} W_{ij} [U_i^\pm (I_j^3 - \frac{3}{2} Y_j) + I_i^\mp V_j^\mp], \\
\bar{V}_\pm &= \sum_i \mu_i V_i^\pm \pm h \sum_{i \neq j} W_{ij} [V_i^\pm (I_j^3 + \frac{3}{2} Y_j) + U_i^\mp I_j^\mp], \\
\bar{I}_3 &= \sum_i \mu_i I_i^3 + h \sum_{i \neq j} W_{ij} [I_i^+ I_j^- - \frac{1}{2} (U_i^+ U_j^- - V_i^+ V_j^-)], \\
\bar{I}_8 &= \sum_i \mu_i Y_i + h \sum_{i \neq j} W_{ij} (U_i^+ U_j^- - V_j^+ V_j^-), \tag{3.4.10}
\end{aligned}$$

where  $\mu_i$  and  $h$  (not Planck constant) are arbitrary parameters. Notice again that the simplest choice of  $W_{ij}$  is given by equation (3.4.8).

When  $i = 1, 2$ ,  $Y(SU(2))$  makes transition between spin singlet and triplet. Now  $Y(SU(3))$  transits  $SU(3)$  singlet and Octet. For instance, setting

$$\begin{aligned}
|\pi^-\rangle &= |d\bar{u}\rangle, \quad |\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle), \quad |K^-\rangle = |d\bar{u}\rangle, \quad |K^0\rangle = |d\bar{s}\rangle, \\
|\eta^0\rangle &= \frac{1}{\sqrt{6}}(-|u\bar{u}\rangle - |d\bar{d}\rangle + 2|s\bar{s}\rangle), \quad |\eta^{0'}\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle). \tag{3.4.11}
\end{aligned}$$

Special interest is the following. When

$$\mu_1 - \mu_2 = -3h, \quad f = -\frac{1}{2}(\mu_1 - \mu_2), \tag{3.4.12}$$

by acting the Yangian operators on the Octet of  $SU(3)$ , we obtain (see Figure 1)

$$\begin{aligned}
\bar{I}_- |\pi^+\rangle &= \frac{1}{\sqrt{6}}(\mu_1 - \mu_2)|\eta^0\rangle + \frac{1}{\sqrt{2}}(\mu_1 + \mu_2)|\pi^0\rangle - \frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'}\rangle, \\
\bar{U}_+ |\bar{K}^0\rangle &= \frac{1}{\sqrt{6}}(\mu_1 + 2\mu_2)|\eta^0\rangle + \frac{1}{\sqrt{2}}\mu_1|\pi^0\rangle - \frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'}\rangle, \\
\bar{U}_- |K^0\rangle &= \frac{1}{\sqrt{6}}(2\mu_1 + \mu_2)|\eta^0\rangle + \frac{1}{\sqrt{2}}\mu_2|\pi^0\rangle + \frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'}\rangle, \\
\bar{V}_+ |K^+\rangle &= \frac{1}{\sqrt{6}}(2\mu_1 + \mu_2)|\eta^0\rangle - \frac{1}{\sqrt{2}}\mu_2|\pi^0\rangle + \frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'}\rangle, \\
\bar{V}_- |K^-\rangle &= -\frac{1}{\sqrt{6}}(\mu_1 + 2\mu_2)|\eta^0\rangle + \frac{1}{\sqrt{2}}\mu_1|\pi^0\rangle + \frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'}\rangle, \\
\bar{I}_3 |\pi^0\rangle &= -\frac{1}{2\sqrt{3}}(\mu_1 - \mu_2)|\eta^0\rangle + \frac{1}{\sqrt{6}}(\mu_1 - \mu_2 + 3h)|\eta^{0'}\rangle, \\
\bar{I}_8 |\eta^0\rangle &= -\frac{1}{3}(\mu_1 - \mu_2)|\eta^0\rangle - \frac{\sqrt{2}}{3}(\mu_1 - \mu_2 + 3h)|\eta^{0'}\rangle, \tag{3.4.13}
\end{aligned}$$

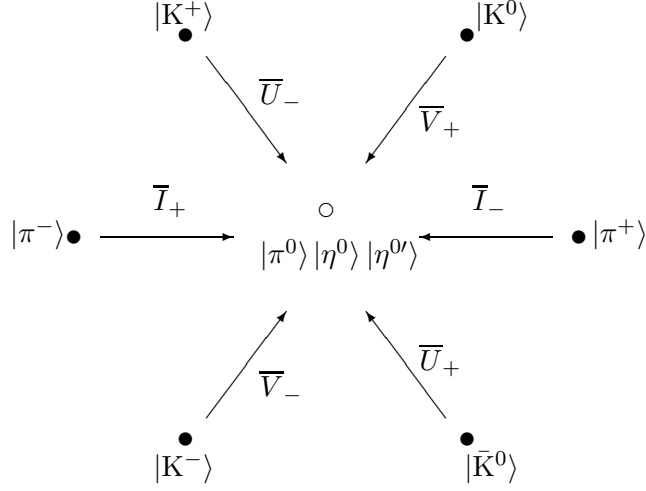


Figure 1: Representation of  $SU(3)$

i.e.,

$$\begin{aligned}
(\bar{I}_{\pm} + fI_{\pm})|\eta^{0'}> &= \pm 2\sqrt{3}h|\pi^{\pm}>, \quad (\bar{U}_{+} + fU_{+})|\eta^{0'}> = -2\sqrt{3}h|K^0>, \\
(\bar{U}_{-} + fU_{-})|\eta^{0'}> &= 2\sqrt{3}h|\bar{K}^0>, \quad (\bar{V}_{\pm} + fV_{\pm})|\eta^{0'}> = -2\sqrt{3}h|K^{\mp}>, \\
(\bar{I}_3 + fI_3)|\eta^{0'}> &= -\sqrt{6}h|\pi^0>, \quad (\bar{I}_8 + fI_8)|\eta^{0'}> = 2\sqrt{2}h|\eta^0>,
\end{aligned} \tag{3.4.14}$$

and

$$\begin{aligned}
(\bar{I}_{\pm} + fI_{\pm})|\pi^{\mp}> &= \pm \sqrt{\frac{3}{2}}h|\eta^0>, \\
(\bar{U}_{+} + fU_{+})|K^0> &= -\frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0> - |\eta^0>), \\
(\bar{U}_{-} + fU_{-})|K^0> &= \frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0> - |\eta^0>), \\
(\bar{V}_{\pm} + fV_{\pm})|K^{\pm}> &= -\frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0> + |\eta^0>), \\
(\bar{I}_3 + fI_3)|\pi^0> &= \sqrt{\frac{3}{2}}h|\eta^0>, \quad (\bar{I}_8 + fI_8)|\eta^0> = \sqrt{3}h|\eta^0>.
\end{aligned} \tag{3.4.15}$$

The Yangian operators play the role to transit the Octet states to the singlet state of  $SU(3)$ .

Whereas, if

$$\mu_1 - \mu_2 = 3h, \quad f = -\frac{1}{2}(\mu_1 + \mu_2), \tag{3.4.16}$$

with the notations

$$(\bar{A}^{(2)} + fA^{(1)})|\eta^{0'}> = 0, \quad A = I_{\alpha}, \quad (\alpha = \pm, 3, 8), \quad U_{\pm}, \quad V_{\pm}, \tag{3.4.17}$$

we have

$$\begin{aligned}
(\bar{I}_\pm + fI_\pm)|\pi^\mp > &= \mp\sqrt{\frac{3}{2}}h|\eta^0 > \pm 2\sqrt{3}h|\eta^{0'} >, \\
(\bar{U}_+ + fU_+)|\bar{K}^0 > &= \frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0 > -|\eta^0 >) - 2\sqrt{3}h|\eta^{0'} >, \\
(\bar{U}_- + fU_-)|K^0 > &= -\frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0 > -|\eta^0 >) + 2\sqrt{3}h|\eta^{0'} >, \\
(\bar{V}_\pm + fV_\pm)|K^\pm > &= \frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0 > +|\eta^0 >) + 2\sqrt{3}h|\eta^{0'} >, \\
(\bar{I}_3 + fI_3)|\pi^0 > &= -\frac{\sqrt{3}}{2}h|\eta^0 > + \sqrt{6}h|\eta^{0'} >, \\
(\bar{I}_8 + fI_8)|\eta^0 > &= h|\eta^0 > - 2\sqrt{2}h|\eta^{0'} >.
\end{aligned} \tag{3.4.18}$$

Obviously, in this case the Yangian operators make the transition from the Octet to a “combined” singlet state of  $SU(3)$ .

### 3.5 $\mathbf{J}^2$ as a new quantum number

Because  $[\mathbf{I}^2, \mathbf{J}^2] = 0$ ,  $[\mathbf{I}^2, I_z] = 0$ ,  $[\mathbf{J}^2, I_z] = 0$ , but  $[\mathbf{J}^2, J_z] \neq 0$ , we can take  $\{\mathbf{I}^2, I_z, \mathbf{J}^2\}$  as a conserved set.

First we consider the case  $\mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{S}_3$ , where  $S_1 = S_2 = S_3 = \frac{1}{2}$ . We shall show that instead of 6-j coefficients and Young diagrams,  $\mathbf{J}^2$  can be viewed as a “collective” quantum number that describes the “history” besides  $\mathbf{S}^2$  ( $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$ ) and  $S_z$ .

As representations of Lie algebra  $SU(2)$ , we have

$$\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \frac{1}{2} = (1 \oplus 0) \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1'}{2}. \tag{3.5.1}$$

Noting that  $|\frac{1}{2}\rangle$  and  $|\frac{1'}{2}\rangle$  are degenerate regarding the total spin  $\frac{1}{2}$ . The usual Lie algebraic base can be easily written as

$$\begin{aligned}
\phi_{\frac{3}{2}, \frac{3}{2}} &= |\uparrow\uparrow\uparrow\rangle, \\
\phi_{\frac{3}{2}, \frac{1}{2}} &= \frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle), \\
\phi_{\frac{3}{2}, -\frac{1}{2}} &= \frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle), \\
\phi_{\frac{3}{2}, -\frac{3}{2}} &= |\downarrow\downarrow\downarrow\rangle,
\end{aligned} \tag{3.5.2}$$

and the two degeneracy states with respect to  $\mathbf{S}^2$  and  $S_z$  are given by:

$$\phi'_{\frac{1}{2}, \frac{1}{2}} = \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle),$$

$$\begin{aligned}
\phi'_{\frac{1}{2}, -\frac{1}{2}} &= \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle - 2|\downarrow\downarrow\uparrow\rangle), \\
\phi_{\frac{1}{2}, \frac{1}{2}} &= \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle), \\
\phi_{\frac{1}{2}, -\frac{1}{2}} &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle).
\end{aligned} \tag{3.5.3}$$

To distinguish  $\phi'$  from  $\phi$  we introduce  $\mathbf{J}$ :

$$\mathbf{J} = \sum_{i=1}^3 u_i \mathbf{S}_i + i\hbar \sum_{i<j}^3 (\mathbf{S}_i \times \mathbf{S}_j), \tag{3.5.4}$$

and calculate  $\mathbf{J}^2$ . It turns out that

$$\begin{aligned}
\mathbf{J}^2 \phi_{\frac{3}{2}, m} &= \left[ \frac{3}{4}(u_1^2 + u_2^2 + u_3^2) + \frac{1}{2}(u_1 u_2 + u_2 u_3 + u_1 u_3) - h^2 \right] \Phi_{\frac{3}{2}, m}; \\
\mathbf{J}^2 \phi'_{\frac{1}{2}, m} &= \left[ \frac{3}{4}(u_1^2 + u_2^2 + u_3^2) + \frac{1}{2}u_1 u_2 - u_2 u_3 - u_1 u_3 - \frac{7}{4}h^2 \right] \Phi'_{\frac{1}{2}, m} \\
&\quad - \frac{\sqrt{3}}{2}(u_1 - u_2 + h)(u_3 + h) \Phi_{\frac{1}{2}, m}; \\
\mathbf{J}^2 \phi_{\frac{1}{2}, m} &= -\frac{\sqrt{3}}{2}(u_1 - u_2 - h)(u_3 - h) \Phi'_{\frac{1}{2}, m} + \left[ \frac{3}{4}(u_1 - u_2)^2 \right. \\
&\quad \left. + \frac{3}{4}u_3^2 - \frac{3}{4}h^2 \right] \Phi_{\frac{1}{2}, m}.
\end{aligned} \tag{3.5.5}$$

In order to make the matrix of  $\mathbf{J}^2$  be symmetric (then it surely can be diagonalized), one should put

$$u_2 = u_1 + u_3. \tag{3.5.6}$$

The eigenvalues of  $\mathbf{J}^2$  are given by

$$\begin{aligned}
\lambda_{\frac{3}{2}} &= 2u_1^2 + 2u_3^2 + 3u_1 u_3 - h^2, \\
\lambda_{\frac{1}{2}}^{\pm} &= u_1^2 + u_3^2 - \frac{5}{4}h^2 \pm \frac{1}{2}[(2u_1^2 - u_3^2 - h^2)^2 + 3(u_3^2 - h^2)^2]^{\frac{1}{2}}.
\end{aligned} \tag{3.5.7}$$

The eigenstates of  $\mathbf{J}^2$  are the rotation of  $\phi'_{\frac{1}{2}, m}$  and  $\Phi_{\frac{1}{2}, m}$ :

$$\begin{pmatrix} \alpha_{\frac{1}{2}, m}^{+} \\ \alpha_{\frac{1}{2}, m}^{-} \end{pmatrix} = \begin{pmatrix} \cos \frac{\varphi}{2} & -\sin \frac{\varphi}{2} \\ \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} \phi'_{\frac{1}{2}, m} \\ \phi_{\frac{1}{2}, m} \end{pmatrix}, \quad \mathbf{J}^2 \alpha_{\frac{1}{2}}^{\pm} = \lambda_{\frac{1}{2}}^{\pm} \alpha_{\frac{1}{2}, m}^{\pm}, \tag{3.5.8}$$

where

$$\sin \varphi = \sqrt{3}(u_3^2 - h^2)/\omega, \quad \omega^2 = (2u_1^2 - u_3^2 - h^2)^2 + 3(u_3^2 - h^2)^2. \tag{3.5.9}$$

It is worth noting that the conclusion is independent of the order, say,  $(\frac{1}{2} \otimes \frac{1}{2}) \otimes \frac{1}{2}$ ,  $\frac{1}{2} \otimes (\frac{1}{2} \otimes \frac{1}{2})$  and the other way. The difference is only in the value of  $\varphi$ .

The above example can be generalized to  $\mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{L}$  where  $S_1 = S_2 = \frac{1}{2}$  and  $\mathbf{L}^2 = l(l+1)$ . As representations of Lie algebra  $SU(2)$ , we have

$$\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes l = (1 \oplus 0) \otimes l = l+1 \quad \begin{matrix} l & l-1 \\ & l \end{matrix} \quad (3.5.10)$$

There are no degeneracy for  $l \pm 1$ , but two  $l$  states can be distinguished in terms of  $\mathbf{J}^2$

$$\begin{aligned} \mathbf{J}^2 \Phi_{l+1,m} &= \left\{ \frac{3}{4}(u_1^2 + u_2^2) + l(l+1)u_3^2 + \frac{1}{2}u_1u_2 + l(u_2u_3 + u_1u_3) \right. \\ &\quad \left. - h^2[l(l+1) + \frac{1}{4}] \right\} \Phi_{l+1,m}, \\ \mathbf{J}^2 \Phi_{l-1,m} &= \left\{ \frac{3}{4}(u_1^2 + u_2^2) + l(l+1)u_3^2 + \frac{1}{2}u_1u_2 - (l+1)u_1u_3 - (l+1)u_2u_3 \right. \\ &\quad \left. - h^2[l(l+1) + \frac{1}{4}] \right\} \Phi_{l-1,m}, \\ \mathbf{J}^2 \Phi_{l,m}^1 &= \left\{ \frac{3}{4}(u_1^2 + u_2^2) + l(l+1)u_3^2 + \frac{1}{2}u_1u_2 - u_2u_3 - u_1u_3 \right. \\ &\quad \left. - 2h^2[l(l+1)\frac{1}{8}] \right\} \Phi_{l,m}^1 - \sqrt{l(l+1)}(u_1 - u_2 + h)(u_3 + h)\Phi_{l,m}^2, \\ \mathbf{J}^2 \Phi_{l,m}^2 &= -\sqrt{l(l+1)}(u_1 - u_2 - h)(u_3 - h)\Phi_{l,m}^1 \\ &\quad + \left[ \frac{3}{4}(u_1 - u_2)^2 + l(l+1)u_3^2 - \frac{3}{4} \right] \Phi_{l,m}^2. \end{aligned} \quad (3.5.11)$$

Again in order to guarantee the symmetric form of the matrix we put

$$u_2 = u_1 + u_3, \quad (3.5.12)$$

then the eigenvalues and eigenstates of  $\mathbf{J}^2$  are given by

$$\lambda_l^\pm = u_1^2 + [l(l+1) + \frac{1}{4}]u_3^2 - h^2[l(l+1) + \frac{1}{2}] \pm \frac{1}{2}\sqrt{P}, \quad (3.5.13)$$

$$\begin{pmatrix} \alpha_{l,m}^+ \\ \alpha_{l,m}^- \end{pmatrix} = \begin{pmatrix} \cos \frac{\varphi}{2} & -\sin \frac{\varphi}{2} \\ \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} \Phi_{l,m}^1 \\ \Phi_{l,m}^2 \end{pmatrix}, \quad (3.5.14)$$

where

$$\omega^2 = P = [2u_1^2 - u_3^2 - h^2(2l(l+1) - \frac{1}{2})]^2 + 4l(l+1)(u_3^2 - h^2)^2, \quad (3.5.15)$$

$$\sin \varphi = \frac{2\sqrt{l(l+1)}}{\omega}(u_3^2 - h^2). \quad (3.5.16)$$

As a simple example, we consider the spin structure of rare gas

$$H = -a\mathbf{L} \cdot \mathbf{S}_1 - b\mathbf{S}_1 \cdot \mathbf{S}_2, \quad (\lambda = \frac{b}{a}). \quad (3.5.17)$$



It describes the interaction of spin  $\mathbf{S}_1$  of an electron excited from  $l$ -shell and the left hole  $\mathbf{S}_2$ .

$$\begin{aligned} H\Phi_{l+1,m} &= -\frac{1}{2}(al + \frac{1}{2}b)\Phi_{l+1,m}, \\ H\Phi_{l-1,m} &= \frac{1}{2}[(l+1)a - \frac{1}{2}b]\Phi_{l-1,m}, \\ H\begin{bmatrix} \Phi_{l,m}^+ \\ \Phi_{l,m}^2 \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} (a - \frac{1}{2}b) & a\sqrt{l(l+1)} \\ a\sqrt{l(l+1)} & \frac{3}{2}b \end{bmatrix}\begin{bmatrix} \Phi_{l,m}^1 \\ \Phi_{l,m}^2 \end{bmatrix}. \end{aligned} \quad (3.5.18)$$

The eigenstates of  $H$  associated to  $l, m$  are

$$\begin{pmatrix} \alpha_{l,m}^+ \\ \alpha_{l,m}^- \end{pmatrix} = \begin{pmatrix} \cos \frac{\varphi}{2} & -\sin \frac{\varphi}{2} \\ \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} \Phi_{l,m}^1 \\ \Phi_{l,m}^2 \end{pmatrix}. \quad (3.5.19)$$

where

$$\sin \varphi = \frac{\sqrt{l(l+1)}}{\omega}, \quad \omega^2 = (\frac{1}{2} - \lambda)^2 + l(l+1), \quad \lambda = \frac{b}{a}. \quad (3.5.20)$$

The eigenvalues are

$$\begin{aligned} \lambda_{l+1} &= -\frac{1}{2}(la + \frac{b}{2}), \quad \lambda_{l-1} = \frac{1}{2}[(l+1)a - \frac{b}{2}]; \\ \lambda_l^\pm &= \frac{1}{4}(a+b) \pm \frac{1}{2}[l(l+1)a^2 + (\frac{a}{2} - b)^2]^{\frac{1}{2}}. \end{aligned} \quad (3.5.21)$$

The rotation should be made in such a way that

$$[H, \mathbf{J}^2] = 0 \quad (3.5.22)$$

which is satisfied if the matrix  $\mathbf{J}^2$  is symmetric, i.e.,

$$\gamma = \frac{\{2u_1^2 - 2h^2[l(l+1) + \frac{1}{4}]\}}{(u_3^2 - h^2)} = 2(1 - \lambda). \quad (3.5.23)$$

Therefore, the parameter  $\gamma$  in  $Y(SU(2))$  determines the rotation angle  $\varphi$ . It is reasonable to think that the appearance of “rotation” of degenerate states is closely related to the “quantum number” of  $\mathbf{J}^2$ . Transition between  $\alpha_{l,m}^+$  and  $\alpha_{l,m}^-$  ( $l=1$ ) can be made by  $J_3$ . Because there are two independent parameters  $u_1$  and  $u_3$  in  $\mathbf{J}$ , one can choose a suitable relation between  $u_3$  and  $\lambda = \frac{b}{a}$  such that

$$J_3\alpha^+ \sim \alpha^-, \quad (3.5.24)$$

i.e., the transition between two degenerate states in Lie-algebra is made through  $J_3$  operator, because of

$$[\mathbf{J}^2, J_3] \neq 0. \quad (3.5.25)$$

### 3.6 Happer degeneracy

In the experiment for  $^{87}\text{Rb}$  molecular there appears new degeneracy ([41]) at the special  $\pm B_0$  (magnetic field), i.e., the Zeeman effect disappears at  $\pm B_0$ . The model Hamiltonian reads ([42]) ( $x$  is scaled magnetic field)

$$H = \mathbf{K} \cdot \mathbf{S} + x(k + \frac{1}{2})S_z, \quad (3.6.1)$$

where  $\mathbf{K}$  is angular momentum and  $\mathbf{K}^2 = K(K+1)$ . It only occurs for spin  $S = 1$ . It turns out that when  $x = \pm 1$  there appears the curious degeneracy, that is, there is a set of eigenstates corresponding to

$$E = -\frac{1}{2}. \quad (3.6.2)$$

The conserved set is  $\{\mathbf{K}^2, G_z = K_z + S_z\}$ . For  $\mathbf{G} = \mathbf{K} + \mathbf{S}$  we have  $G = k \pm 1, k$ . The eigenstates are specified in terms of three families:  $T, B$  and  $D$ . Only D-set possesses the degeneracy.

Happer gives, for example, the eigenstates for  $x = \pm 1$  ([42]):

$$\begin{aligned} x = +1 & \quad H\alpha_{Dm} = (-\frac{1}{2})\alpha_{Dm}, \\ x = -1 & \quad H\beta_{Dm} = (-\frac{1}{2})\beta_{Dm}, \end{aligned} \quad (3.6.3)$$

and shows that

$$\begin{aligned} \alpha_{Dm} &= [2(K + \frac{1}{2})(K + m + \frac{1}{2})]^{-\frac{1}{2}} \{ -[\frac{(K-m+1)(K+m+1)}{2}]^{\frac{1}{2}}\alpha_1 \\ &\quad + [(K+m)(K+m+1)]^{\frac{1}{2}}\alpha_2 + [\frac{(K-m)(K+m)}{2}]^{\frac{1}{2}}\alpha_3 \}; \end{aligned} \quad (3.6.4)$$

$$\begin{aligned} \beta_{Dm} &= [2(K + \frac{1}{2})(K - m + \frac{1}{2})]^{-\frac{1}{2}} \{ [\frac{(K-m)(K+m)}{2}]^{\frac{1}{2}}\alpha_1 \\ &\quad + [(K-m)(K-m+1)]^{\frac{1}{2}}\alpha_2 - [\frac{(K-m+1)(K+m+1)}{2}]^{\frac{1}{2}}\alpha_3 \}, \end{aligned} \quad (3.6.5)$$

where  $\alpha_1 = e_1 \otimes e_{m-1}$ ,  $\alpha_2 = e_0 \otimes e_m$  and  $\alpha_3 = e_{-1} \otimes e_{m+1}$ .

It is natural to ask what is the transition operator between  $\alpha_{Dm}$  and  $\beta_{Dm}$ ? The answer is Yangian operator. In fact, introducing

$$J_{\pm} = aS_{\pm} + bK_{\pm} \pm (s_{\pm}K_z - s_zK_{\pm}), \quad (3.6.6)$$

we find that by choosing  $a = -\frac{k+1}{2}, b = 0$ , we have

$$\beta_{Dm} \xrightarrow{J_+} \lambda_1(m)\alpha_{Dm+1} \quad \text{and} \quad \alpha_{Dm} \xrightarrow{J_-} \lambda_2(m)\beta_{Dm-1}; \quad (3.6.7)$$

and by choosing  $a = \frac{k}{2}, b = 0$ , we have

$$\beta_{Dm} \xrightarrow{J_-} \lambda'_1(m) \alpha_{Dm-1} \quad \text{and} \quad \alpha_{Dm} \xrightarrow{J_+} \lambda'_2(m) \beta_{Dm+1}. \quad (3.6.8)$$

The Yangian makes the transition between the states with  $B$  and  $-B$ , which here is only for  $S = 1$ . The reason is that for  $S = 1$  there are two independent coefficients in the combination of  $\alpha_1, \alpha_2$  and  $\alpha_3$  and there are two free parameters in  $\mathbf{J}$ . Hence the number of equations are equal to those of free parameters ( $a$  and  $b$ ), so we can find a solution. The numerical computation shows that only  $S = 1$  gives rise to the new degeneracy ([42]) that prefers the Yangian operation ([43]).

### 3.7 New degeneracy of extended Breit-Rabi Hamiltonian

As was shown in the Happer's model ( $H = \mathbf{K} \cdot \mathbf{S} + x(k + \frac{1}{2})S_3$ ) there appeared new degeneracy for  $S = 1$ . It has been pointed out that the above degeneracy with respect to Zeeman effect cannot appear for  $\text{spin} = \frac{1}{2}$ . Actually, in this case it yields for  $S = \frac{1}{2}$  ([42]),

$$E = -\frac{1}{4} - \omega_m S_3, \quad (3.7.1)$$

where

$$\omega_m^2 = [(1 + x^2)(k + \frac{1}{2}) + 2xm](k + \frac{1}{2}). \quad (3.7.2)$$

Therefore if the Happer's type of degeneracy can occurs, there should be  $\omega_m = 0$  that means

$$x_0 = -\frac{m}{k} \pm i\sqrt{1 - \frac{m^2}{k^2}} \quad (k = K + \frac{1}{2}), \quad (3.7.3)$$

i.e., the magnetic field should be complex.

However, the situation will be completely different, if a third spin is involved. For simplicity we assume  $S_1 = S_2 = S_3 = \frac{1}{2}$  in the Hamiltonian:

$$H = -(a\mathbf{S}_2 + b\mathbf{S}_3) \cdot \mathbf{S}_1 + x\sqrt{ab}S_1^z, \lambda = b/a, \quad (3.7.4)$$

then besides two non-degenerate states, there appears the degenerate family:

$$H\alpha_{D,\pm\frac{1}{2}}^\pm = -(\frac{a+b}{4})\alpha_{D,\pm\frac{1}{2}}^\pm, \quad \text{for } x = \pm 1, \quad (3.7.5)$$

where

$$\alpha_{D,+ \frac{1}{2}}^\pm = -\sqrt{2}\lambda |\uparrow\uparrow\downarrow\rangle \pm \sqrt{\lambda} |\uparrow\downarrow\uparrow\rangle + (1 \pm \sqrt{\lambda}) |\downarrow\uparrow\uparrow\rangle; \quad (3.7.6)$$

$$\alpha_{D,-\frac{1}{2}}^{\pm} = -\sqrt{2}\lambda |\downarrow\downarrow\uparrow\rangle \mp \sqrt{\lambda} |\downarrow\uparrow\downarrow\rangle + (1 \mp \sqrt{\lambda}) |\uparrow\downarrow\downarrow\rangle. \quad (3.7.7)$$

The expecting value of  $S_1^z$  are

$$\langle \alpha_{D,\pm\frac{1}{2}}^+ | S_1^z | \alpha_{D,\pm\frac{1}{2}}^+ \rangle \sim \sqrt{\lambda} \quad (x = 1); \quad (3.7.8)$$

$$\langle \alpha_{D,\pm\frac{1}{2}}^- | S_1^z | \alpha_{D,\pm\frac{1}{2}}^- \rangle \sim -\sqrt{\lambda} \quad (x = -1). \quad (3.7.9)$$

namely, at the special magnetic field ( $x = \pm 1$ ) the observed  $\langle S_1^z \rangle$  still opposite to each other for  $x = \pm 1$ , but without the usual Zeeman split.

The reason of the appearance of the new degeneracy is obvious. The two spins  $\mathbf{S}_2$  and  $\mathbf{S}_3$  here play the role of  $S = 1$  in comparison with Happer model.

### 3.8 Super Yang-Mills ( $N = 4$ )-Lipatov model and $Y(SO(6))$

Beisert et al([44-45]), Dolan-Nappi-Witten (DNW,[34]) and other authors ([46-47]) proposed to take the quantum correction of the dilatation operator  $\delta D$  ( $D \in SO(4,2)$  is a subalgebra of  $PSU(2,2|4)$ ) as Hamiltonian for super Yang-Mills ( $N = 4$ ):

$$H = \sum_{\alpha} H_{\alpha\alpha+1}, \quad (3.8.1)$$

$$H_{\alpha\alpha+1} = 2 \sum_j h(j) P_{\alpha\alpha+1}^j, \quad h(j) = \sum_{k=1}^j \frac{1}{k}, \quad h(0) = 1. \quad (3.8.2)$$

where  $P^j$  is projector for the weight  $j$  of  $SU(2)$  and  $\alpha$  stands for “lattice” index. DNW showed that ([34])

$$[H, Y(SO(6))] = 0. \quad (3.8.3)$$

It turns out that the Hamiltonian  $H$  is nothing but Lipatov model ([48]) which was related to the Yang-Baxter form by Lipatov ([49]), Faddeev and Korchemsky ([50]).

Based on Tarasov, Takhtajan and Faddeev([51]) the  $\check{R}$ -matrix associated with any spin  $S$  reads

$$\check{R}(u) = \frac{\Gamma(u-s)\Gamma(u+2s+1)}{\Gamma(u-\hat{J})\Gamma(u+\hat{J}+1)}, \quad (3.8.4)$$

where  $u$  is spectrum parameter and  $s$  the spin (arbitrary). The trigonometric Yang-Baxterization ([52]) gives

$$\check{R}(u) = \sum_{j=0} \rho_j(x) P_j(q) \quad (x = e^{iu}), \quad (3.8.5)$$

where  $P_j(q)$  is the  $q$ -deformed product with weight  $j$ . Taking the rational limit ([9],[36]) we have

$$\rho_j \Rightarrow \frac{\Gamma(u)\Gamma(u+1)}{\Gamma(u-j)\Gamma(u+j+1)}, \quad P_j(q) \Rightarrow P_j. \quad (3.8.6)$$

The Hamiltonian for the lattices  $\alpha$  and  $\alpha+1$

$$H_{\alpha\alpha+1} = I_1 \times I_2 \times \cdots \times I_{\alpha-1} \times \frac{d}{du} \check{R}(u)|_{u=0} [\check{R}(0)]^{-1} \times I_{\alpha+2} \times \cdots \quad (3.8.7)$$

is then

$$H = \sum_{\alpha} H_{\alpha\alpha+1} \quad (3.8.8)$$

where

$$\begin{aligned} H_{\alpha\alpha+1} &= \{-\psi(-\hat{J}_{\alpha\alpha+1}) - \psi(\hat{J}_{\alpha\alpha+1} + 1) + \psi(1+2s) + \psi(1-2s) - \frac{1}{2s}\}_{|s=0} \\ &= \sum_j \{-\psi(-j) - \psi(j+1) + 2\psi(1) - \lim_{x \rightarrow 0} \frac{1}{x}\} P_{\alpha\alpha+1}^j. \end{aligned} \quad (3.8.9)$$

It describes the QCD correction to the parton model shown by Lipatov ([48-49]). The diagonalization of Lipatov model has probably been achieved by de Vega and Lipatov ([53-54]). Noting that the  $j$  indicates the block in the reducible block-diagonal form.

Using

$$\begin{aligned} \psi(x+1) &= \psi(x) + \frac{1}{x}, \\ \psi(x+n) &= \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}, \\ \psi(1) &= -c, \end{aligned} \quad (3.8.10)$$

and hence

$$\begin{aligned} \psi(j+1) &= \psi(1) + \sum_{k=1}^j \frac{1}{k} = \psi(1) + h(j) \\ \psi(-j) &= \psi(1) + h(j) - \lim_{x \rightarrow 0} \frac{1}{x}. \end{aligned} \quad (3.8.11)$$

We obtain

$$H_{\alpha,\alpha+1} = (-2) \sum_j h(j) P_{\alpha\alpha+1}^j. \quad (3.8.12)$$

Separating the finite part from the infinity the  $H$  is nothing but the  $\delta D$  derived in super Yang-Mills ( $N=4$ ) with the approximation. Of course, the derivation of  $\delta D$  based on super Yang-Mills ( $N=4$ ) explores much larger symmetry than Lipatov model. Therefore, DNW's result shows that the Lipatov's model possesses  $Y(SO(6))$  symmetry.

To obtain  $Y(SO(6))$  in terms of RTT relation we start from the rational solution of  $\check{R}$ -matrix whose general form for  $O(N)$  was firstly by Zamolodchikov and Zamolodchikov ([35]) and extended through rational limit of trigonometric Yang-Baxterization ([36]):

$$\check{R} = u[u - \frac{1}{2}(N-2)a]P + \alpha u A_N + [-u\alpha + \frac{\alpha^2}{2}(N-2)]I. \quad (3.8.13)$$

where  $u$  is spectrum parameter and  $\alpha$  a free parameter allowed by YBE. Here we adopt the convention of Jimbo:

$$P_{cd}^{ab} = \delta_d^a \delta_c^b, \quad (A_N)_{cd}^{ab} = \delta^{a,-b} \delta_{c,-d} \quad (3.8.14)$$

where

$$a, b, c, d = [-(\frac{N-1}{2}), -(\frac{N-1}{2}) + 1, \dots, (\frac{N-1}{2})] \quad (3.8.15)$$

and  $N = 2n + 1$  for  $B_n$  and  $N = 2n$  for  $C_n, D_n$ .

The R-matrix is given by

$$R = \check{R}P = u(u - 2\alpha)I + u(2u - \alpha)P + 2u\alpha A_N, \quad (3.8.16)$$

that coincides with Zamolodchikov's  $S$ -matrix (up to an over all factor considering the CDD poles) with  $\alpha = 1$  and  $u = \frac{\theta}{i\lambda}$ . Actually, Zamolodchikov's  $S$ -matrix is universal, i.e., model independent.

$$\begin{aligned} S(\theta) = R(u) &= Q^\pm(u)u(u-2)[I + \frac{\sigma_3}{\sigma_2}P + \frac{\sigma_1}{\sigma_2}A_N] \\ &= Q^\pm(u)u(u-2)[I - \frac{1}{u}P + \frac{2}{u-2}A_N], \\ Q^\pm(u) &= \frac{\Gamma(\pm\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi})\Gamma(\frac{1}{2} - i\frac{\theta}{2\pi})}{\Gamma(\frac{1}{2} \pm \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi})\Gamma(-i\frac{\theta}{2\pi})} \end{aligned} \quad (3.8.17)$$

where  $\lambda = \frac{2\pi}{N-2}$ ,  $\theta = i\lambda u$ . The spectrum parameter  $u$  is one-dimensional, but  $u$  can be taken to be the cut-off in 4-dimensional quantum field theory, for example

$$u \sim \ln \Lambda^2, \quad (3.8.18)$$

where  $\Lambda^2$  is Lorentz invariant, i.e., scalar. This is the reason why asymptotic behavior of quantum field theory model may be related to Yang-Baxter system. The Bethe Ansatz for  $S(\theta)$  with  $SO(6)$  was discussed by Minahan and Zarembo ([46]).

For given  $\check{R}(u)$  one can easily obtain Hamiltonian by

$$H = [\frac{\partial \check{R}(u)}{\partial u} \check{R}(u)]|_{u=0}, \quad (3.8.19)$$

for  $O(N)$ .

However, the essential connection between Lipatov model and  $SO(6)$ -RTT formulation is still missing.

## 4 Remarks

Although there has been certain progress of Yangian's application in physics, there are still open questions:

- (1) How can the Yangian representations help to solve physical models, in particular, in strong correlation models?
- (2) Direct evidences of Yangian in the real physics.
- (3) What is the geometric meaning of Yangian?

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